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Potential systems for PDEs having several conservation laws

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Abstract A generalization of the usual procedure for constructing potential systems for systems of partial differential equations with multidimensional spaces of conservation laws is considered. More precisely, for the construction of potential systems with a multi-dimensional space of local conservation laws, instead of using only basis conservation laws, their arbitrary linear combinations are used that are inequivalent with respect to the equivalence group of the class of systems or symmetry group of the fixed system. It appears that the basis conservation laws can be equivalent with respect to groups of symmetry or equivalence transformations, or vice versa; in this sense the number of independent linear combinations of conservation laws can be grater than the dimension of the space of conservation laws. The first possibility leads to an unnecessary, often cumbersome, investigation of equivalent systems, the second one makes possible that a great number of inequivalent potential systems are missed. Examples of all these possibilities are given.

Keywords Conservation law · Equivalence transformation · Potential symmetry

1 Introduction

When one investigates a physical process, knowledge of symmetry properties of the modelling system of PDEs can be very useful for understanding the behavior of solutions of the model. Thus, e.g., a group-invariance property of a system of differential equations allows to generate new solutions from the known ones, to construct conservation laws and to find wide classes of exact invariant solutions. This becomes especially important in the investigation of nonlinear models, where every single partial solution plays an important role. Even if it does not solve any real boundary-value problem, it can be used, e.g., as a testing solution for different numeric or approximate algorithms. Moreover, for many nonlinear systems, invariant solutions are the only known solutions. This is just one reason why any new symmetry is of great importance for systems of PDEs. In this short note we illustrate a way of constructing a larger number of potential systems that can lead to finding new potential symmetries.

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The concept of potential symmetry was introduced by Bluman et al. [1, Chapter 7], [2] in the late 80s; see also the related notion of quasi-local symmetry [3,4]. Namely, if at least one of equations of a system of PDEs can be written in conserved form, then using it one can introduce potential variable(s). Attaching equations containing the new potential variable(s) to the system, one obtains a new (potential) system, nonlocally related to the initial one. Any symmetry of the potential system induces symmetry of the initial system. (Generally speaking, the inverse statement is a bit different: symmetries of the initial system induce symmetries of potential systems or equivalence transformations in the set of potential systems corresponding to the initial system [5].) If the symmetry transformations of the local variables depend explicitly on the potential variable(s), the obtained symmetry projects to a nonlocal one for the initial system and is called *potential symmetry*.

The above procedure of finding potential symmetries has been generalized in [5] by admitting dependence of symmetries on potentials associated to several conservation laws simultaneously. Below we will use the attribute "simplest" to emphasize that the potential system is constructed using a single conservation law only. Here we use a slightly generalized procedure of constructing potential systems for systems of PDEs admitting multi-dimensional spaces of conservation laws.

Previously, for the construction of potential symmetries only basis conservation laws were used. However, this does not guarantee that the obtained potential systems (and therefore, the obtained potential symmetries) will be inequivalent with respect to a local symmetry group of the initial system or local equivalence group of the class of the system. This may lead to an unnecessarily complicated investigation of equivalent systems with, in fact, no new result. Or vice versa, there exists a linear combination of basis conservation laws that leads to a potential system inequivalent to "basis" ones. In such a case, when considering only potential systems constructed with basis conservation laws one can "miss" some of the potential symmetries.

2 Basic notions on conservation laws and potential symmetry

For simplicity we consider here the case of the simplest potential systems for systems of $(1 + 1)$-dimensional equations only. Note that all statements and notions below can be easily generalized to the n-dimensional case and to general potential systems constructed using several conservation laws simultaneously.

Let $\mathcal{L}$ be a system $L(t, x, u_{(\rho)}) = 0$ of l PDEs $L^1 = 0, \ldots, L^l = 0$ for the unknown functions $u = (u^1, \ldots, u^m)$ of the independent variables t and x. Here $u_{(\rho)}$ denotes the set of all partial derivatives of the functions u of order not greater than ρ, including u as the derivatives of zeroth order.

A *conservation law* of the system $\mathcal{L}$ can be understood as a divergence expression

$$D_t T(t, x, u_{(r)}) + D_x X(t, x, u_{(r)}) = 0 \tag{1}$$

which vanishes for all solutions of $\mathcal{L}$. Here D_t and D_x are the operators of total differentiation with respect to t and x, respectively. The differential functions T and X are correspondingly called a *density* and a *flux* of the conservation law and the tuple (T, X) is a *conserved vector* of the conservation law.

The crucial notion of the theory of conservation laws is one of equivalence and triviality of conservation laws. Two conserved vectors (T, X) and (T', X') are *equivalent* if there exist functions $\hat{T}$, $\hat{X}$ and H of t, x and derivatives of u such that $\hat{T}$ and $\hat{X}$ vanish for all solutions of $\mathcal{L}$ and $T' = T + \hat{T} + D_x H$, $X' = X + \hat{X} - D_t H$. A conserved vector is called *trivial* if it is equivalent to the zeroth vector.

The notion of linear dependence of conserved vectors is introduced in a similar way. Namely, a set of conserved vectors is *linearly dependent* if a linear combination of them is a trivial conserved vector.

Although in many simple cases conservation laws can be investigated in the above empiric framework, for deeper analysis one often needs to consider more rigorous definitions, that can be found, e.g., in [5–7].

Let the system $\mathcal{L}$ be totally nondegenerate [8, Sects. 2.6 and 4.3]. (Roughly speaking, it can be understood that the system admits a Cauchy–Kovalevskaya formulation.) Then application of the Hadamard lemma to (1) and integrating by parts imply that the left-hand side of any conservation law of $\mathcal{L}$ can always be presented up to the equivalence relation as a linear combination of left-hand sides of independent equations from $\mathcal{L}$ with coefficients λ^μ being functions of t, x and derivatives of u:

$$D_t T + D_x X = \lambda^1 L^1 + \cdots + \lambda^l L^l. \qquad (2)$$

Equation (2) and the l-tuple $\lambda = (\lambda^1, \ldots, \lambda^l)$ are called the *characteristic form* and the *characteristic* of the conservation law $D_t T + D_x X = 0$, respectively.

The characteristic λ is *trivial* if it vanishes for all solutions of $\mathcal{L}$. Since $\mathcal{L}$ is nondegenerate, the characteristics λ and $\tilde{\lambda}$ satisfy (2) for the same conserved vector (T, X) and, therefore, are called *equivalent* iff $\lambda - \tilde{\lambda}$ is a trivial characteristic.

Any conservation law (1) of $\mathcal{L}$ allows us to deduce the new dependent (potential) variable v by means of the equations

$$v_x = T, \quad v_t = -X. \qquad (3)$$

In the case of single equation $\mathcal{L}$, equations of form (3) combine into a complete potential system since $\mathcal{L}$ is a differential consequence of (3). Such systems often admit a number of nontrivial symmetries and so they are of great interest. If the transformation of some of the nonlocal variables t, x or u depends explicitly on variable v, such a symmetry is nonlocal for the initial equation (system) and is called *potential symmetry*.

3 New potential systems

In [7] a new approach of choosing conservation laws for introducing potentials in order to obtain a larger number of inequivalent potential systems, has been proposed. More precisely, for constructing potential systems in cases when the dimension of the space of conservation laws is greater than one, instead of using only basis conservation laws we propose to use their arbitrary linear combinations being inequivalent with respect to an equivalence group of the class of systems or symmetry group of the fixed system. It is apparent that the number of such independent linear combination is not necessarily equal to the dimension of the space of conservation laws, as was always assumed previously. It is possible also that basis conservation laws can be equivalent with respect to groups of symmetry or equivalence transformations or vice versa, the number of independent (in this sense) linear combinations of conservation laws can be greater than the dimension of the space of conservation laws. The second possibility leads to an unnecessary, often cumbersome, study of equivalent systems, while the third causes a great number of inequivalent potential systems to be missed. Below we illustrate all these three possibilities and show an example when such extra potential systems lead to new potential symmetries.

In this sense the most classical is an example of diffusion equations, for which indeed all possible inequivalent potential systems can be constructed using basis conservation laws only.

Example 1 Consider a class of nonlinear diffusion equations of the form

$$u_t = (A(u)u_x)_x, \quad A(u) \neq \text{const}\,. \qquad (4)$$

The equivalence group $G_1^\sim$ of this class consists of the transformations

$$\tilde{t} = \varepsilon_1 t + \varepsilon_4, \quad \tilde{x} = \varepsilon_2 x + \varepsilon_5, \quad \tilde{u} = \varepsilon_3 u + \varepsilon_6, \quad \tilde{A} = \varepsilon_1^{-1}\varepsilon_2^2 A,$$

where $\varepsilon_1, \ldots, \varepsilon_6$ are arbitrary constants, $\varepsilon_1\varepsilon_2\varepsilon_3 \neq 0$.

It is well-known that for arbitrary values of A this equation possesses two linearly independent conservation laws of the form

$$D_t(u) + D_x(-Au_x) = 0 \quad \text{and} \quad D_t(xu) + D_x\left(-xAu_x + \int A\mathrm{d}u\right) = 0.$$

Therefore, the most general form of potential system depending on one potential that can be constructed for (4) has the form

$$v_x = (c_1x + c_2)u, \quad v_t = c_1\left(xAu_x - \int A\mathrm{d}u\right) + c_2Au_x.$$

Depending on the value of c_1 (is it equal to 0 or not), using a translation of x from equivalence group $G^{\sim}$ and a trivial scaling of the potential variable, one can map this general system to one of the following two inequivalent systems:

$$v^1_x = u, \quad v^1_t = Au_x, \quad \text{or}$$
$$v^2_x = xu, \quad v^2_t = xAu_x - \int A\mathrm{d}u.$$

Together with the potential system

$$v^1_x = u, \quad v^1_t = Au_x, \quad v^2_x = xu, \quad v^2_t = xAu_x - \int A\mathrm{d}u,$$

constructed with simultaneous usage of two potentials, they exhaust all possible inequivalent potential systems that can be constructed from local conservation laws of equations of class (4).

In the second example we have a different situation: the number of inequivalent potential systems is less than the dimension of the space of conservation laws.

Example 2 Consider a class of nonlinear telegraph equations of the form

$$x^{-1}u_{tt} = (F(u)u_x)_x + H(u)u_x.$$

It was shown in [9] that it admits a four-dimensional space of first-order conservation laws containing a subspace spanned by

$$D_t(u_t \sin t - u\cos t) - D_x\left(\sin t\left(xFu_x + xu - \int F\mathrm{d}u\right)\right) = 0,$$
$$D_t(u_t \cos t + u\sin t) - D_x\left(\cos t\left(xFu_x + xu - \int F\mathrm{d}u\right)\right) = 0.$$

It is easy to see that under the action of equivalence transformations of a time translation there exists only a single locally inequivalent simplest potential system having the form

$$v_x = u_t \sin t - u\cos t, \quad v_t = \sin t\left(xFu_x + xu - \int F\mathrm{d}u\right).$$

Finally, we consider the most interesting example of the class of wave equations illustrating the possibility of constructing extra potential systems, yielding new potential symmetries.

Example 3 Consider the class of wave equations

$$u_{tt} = (f(u)u_x)_x. \tag{5}$$

Its equivalence group $G^{\sim}_w$ consists of scaling and translation transformations of $t,\ x$ and u.

For this equation the following local conservation laws with characteristics of zero order are known (see, e.g., [10]):

$$D_t(u_t) - D_x(fu_x) = 0, \quad D_t(tu_t - u) - D_x(tfu_x) = 0,$$
$$D_t(xu_t) - D_x\left(xfu_x - \int f\mathrm{d}u\right) = 0, \quad D_t(x(tu_t - u)) - D_x\left(t\left(xfu_x - \int f\mathrm{d}u\right)\right) = 0.$$

Their characteristics are $1,\ t,\ x$ and tx, respectively. Therefore, the most general simplest potential system can be constructed by using a local conservation law having characteristic $c_1tx + c_2x + c_3t + c_4$, where c_i are arbitrary constants.

If $c_1 \neq 0$, then without loss of generality we can assume that $c_1 = 1$. Using equivalence transformations $x \to x - c_3,\ t \to t - c_2$ we can reduce this characteristic to form $xt + c_4$. Applying additionally scaling transformations, we get $xt+\varepsilon$, where $\varepsilon = 0, 1$. Considering similarly the case $c_1 = 0$, we obtain the following inequivalent (in this sense) characteristics: $x+\varepsilon t,\ t$ and 1, where $\varepsilon = 0, 1$. In this way we get the following simplest inequivalent potential systems:

$$v^1_x = u_t, \quad v^1_t = fu_x, \tag{6}$$
$$v^2_x = tu_t - u, \quad v^2_t = tfu_x, \tag{7}$$
$$v^3_x = xu_t + \varepsilon(tu_t - u), \quad v^3_t = xfu_x - \int f\mathrm{d}u + \varepsilon tfu_x, \tag{8}$$
$$v^4_x = x(tu_t - u) + \varepsilon u_t, \quad v^4_t = t\left(xfu_x - \int f\mathrm{d}u\right) + \varepsilon fu_x. \tag{9}$$

A classification of symmetries of the potential systems (6), (7), (8) $|_{\varepsilon=0}$ and (9) $|_{\varepsilon=0}$ is considered in [10], while the potential systems

$$v_x = xu_t + tu_t - u, \quad v_t = xfu_x - \int f\mathrm{d}u + tfu_x, \tag{10}$$
$$v_x = x(tu_t - u) + u_t, \quad v_t = t\left(xfu_x - \int f\mathrm{d}u\right) + fu_x, \tag{11}$$

are new and can lead to new potential symmetries. In particular, system (11) gives potential symmetries for (5) if and only if $f = 1 \mod G^{\sim}_w$. The corresponding potential algebra has the form

$$\Big\langle \frac{1}{x^2-t^2}(t\partial_t - x\partial_x),\ \left(t + \frac{2x}{x^2-t^2}\right)\partial_t + \left(x - \frac{2t}{x^2-t^2}\right)\partial_x,$$
$$-\frac{1}{4}(3tx^2+4x+t^3)\partial_t - \frac{1}{4}(x^3+3t^3x+4t)\partial_x + \left(v + \frac{1}{2}(t^2+x^2)u\right)\partial_u$$
$$+\left((1+2tx+t^2x^2)u - \frac{1}{2}(t^2+x^2)v\right)\partial_v,\ \mu\partial_u + \phi\partial_v\Big\rangle,$$

where $\mu = \mu(t,x)$ and $\phi = \phi(t,x)$ satisfy the following system of linear equations

$$\phi_t = (xt+1)\mu_x - t\mu, \quad \phi_x = (xt+1)\mu_t - x\mu.$$

Similarly one can prove that Eq. 5 with $f = 1$ admits potential symmetries associated with potential system (10).

4 Conclusions

The view proposed in [7] on equivalence of conservation laws gives rise to a generalization of the procedure for the construction of inequivalent potential systems. In this note we have shown how to find an optimal set of inequivalent potential systems associated with the given system of differential equations.

Previously, for the construction of the simplest potential systems, only basis conservation laws were used. It was just luck that for the considered cases they exhaust all possible inequivalent potential systems. This happened because an arbitrary linear combination of their basis conservation laws could be reduced to a subset of the basis elements by means of a symmetry or equivalence transformation of the considered systems. However, often, then the situation can be much more complicated. Thus, the basis conservation laws may be equivalent with respect to groups of symmetry or equivalence transformations, or vice versa, the number of independent linear combinations of conservation laws may be greater than the dimension of the space of conservation laws. Here we included three examples illustrating all these possibilities. In the latter two cases, for the construction of potential systems, instead of using only basis conservation laws, we propose to use their arbitrary linear combinations being inequivalent with respect to an equivalence group of the class of systems or symmetry group of the fixed system.

As one can see from the Examples 2 and 3, using a normal way of constructing potential systems instead of the approach proposed here, can lead to a needlessly cumbersome investigation of equivalent systems with no new results or may cause a great number of inequivalent potential systems and potential symmetries to be missed.

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References

1. Bluman GW, Kumei S (1989) Symmetries and differential equations. Springer, New York
2. Bluman GW, Reid GJ, Kumei S (1988) New classes of symmetries for partial differential equations. J Math Phys 29:806–811
3. Akhatov ISh, Gazizov RK, Ibragimov NKh (1987) Group classification of equation of nonlinear filtration. Dokl AN SSSR 293: 1033–1035
4. Akhatov ISh, Gazizov RK, Ibragimov NKh (1989) Nonlocal symmetries. A heuristic approach, Itogi Nauki i Tekhniki, Current problems in mathematics. Newest results 34:3–83 (Russian, translated in J Soviet Math, 1991, 55:1401–1450)
5. Popovych RO, Ivanova NM (2005) Hierarchy of conservation laws of diffusion-convection equations. J Math Phys 46 (043502, math-ph/0407008)
6. Zharinov VV (1986) Conservation laws of evolution systems. Teoret Mat Fiz 68(2):163–171
7. Ivanova NM, Popovych RO, Sophocleous C Group analysis of variable coefficient diffusion-convection equations. III. Conservation laws, arXiv:0710.3053
8. Olver PJ (1993) Applications of Lie groups to differential equations. Springer-Verlag, New York
9. Huang D-j, Ivanova NM (2007) Group analysis and exact solutions of a class of variable coefficient nonlinear telegraph equations. J Math Phys 48 (073507, 23 p. math-ph/0702072)
10. Bluman G, Cheviakov AF (2007) Nonlocally related systems. Linearization and nonlocal symmetries for the nonlinear wave equation. J Math Anal Appl 333:93–111